Homework 4, due 9/23

1. Let $f : \mathbf{C} \to \mathbf{C}$ be holomorphic, and not constant. Show that $f(\mathbf{C})$ is dense in $\mathbf{C}$.

2. Let f be a meromorphic function on $\mathbf{C}$.

   (i) Suppose that there exist $k, C > 0$ such that $|f(z)| \leq C|z|^k$ for all $|z| > C$. Prove that f is a rational function, i.e. there are polynomials p, q such that $f = p/q$.

   (ii) Suppose that the function $g(w) = f(1/w)$ is also meromorphic on $\mathbf{C}$. Prove that f is a rational function.

3. Find the Laurent series of the function
$$f(z) = \frac{1}{1 - z^2},$$
around the point $z = -1$. Where does the series converge?

4. Let $f : \mathbf{C} \setminus \{0\} \to \mathbf{C}$ be the meromorphic function defined by
$$f(z) = \frac{1 - \cos z}{z^5}.$$
Find $\mathrm{ord}_0(f)$.

5. Prove that the function $f(z) = \sin(1/z)$ has an essential singularity at $z = 0$.

6. Let $f : D(0,1) \to \mathbf{C}$ be holomorphic such that $f(0) = 0$. Show that there is an integer m, an $r > 0$, and a holomorphic $g : D(0,r) \to \mathbf{C}$ with $g(0) \neq 0$ such that for $z \in D(0,r)$ we have
$$f(z) = \big[zg(z)\big]^m.$$

7. Consider the improper integral
$$I = \lim_{R\to\infty} \int_0^R e^{ix^2}\, dx$$
on the positive real axis. Prove that
$$I = \lim_{R\to\infty} \int_{\gamma_R} e^{iz^2}\, dz,$$
where γ_R is the line segment $\gamma_R(t) = te^{i\theta}$ for any $\theta \in (0, \pi/2)$, with $t \in [0, R]$. Deduce that
$$I = e^{\pi i/4} \int_0^\infty e^{-x^2}\, dx = \frac{\sqrt{\pi}}{2} e^{\pi i/4}.$$